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INTEGRAL POINTS ON THE HYPERBOLA $y^2 = 56x^2 + 1$

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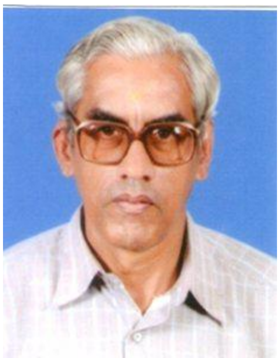

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ABSTRACT

The Binary Quadratic Equation given by $y^2 = 56x^2 + 1$ is analyzed for its patterns of non-zero distinct integral solutions. A few interesting relations between the solutions and special number patterns namely Polygonal number, Pyramidal number and Nasty number are presented. Employing the integral solutions of the equation under consideration, a few patterns of Pythagorean triangles are observed.

KEYWORDS

Integral solutions, Binary Quadratic

MATHEMATICS SUBJECT CLASSIFICATION: 11D09

NOTATIONS

- $t_{m,n}$ - Polygonal number of rank n with size m
- P_n^m - Pyramidal number of rank n with size m
- $Ct_{m,n}$ - Centered Polygonal number of rank n with size m
- S_n - Star number of rank n

INTRODUCTION

The Binary Quadratic Equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different integral values [1, 2, 3, 4]. In [5] infinitely many Pythagorean triangles in each of which hypotenuse is four times the product of the generators added with unity are obtained by employing the non-integral solutions of binary quadratic equation $y^2 = 3x^2 + 1$.In [6], a special Pythagorean triangle is obtained by employing the integral solutions of $y^2 = 10x^2 + 1$.In [7] different patterns of infinitely many Pythagorean triangles are obtained by employing the integral solutions of $y^2 = 12x^2 + 1$. In this context, one may also refer [8-18]. These results have motivated us to search for the integral solutions of yet another binary quadratic equation $y^2 = 56x^2 + 1$. A few interesting properties among the solutions are presented. Employing the integral solutions of the equation under consideration a few patterns of Pythagorean triangles are obtained.

METHOD OF ANALYSIS

Consider the binary quadratic equation

$$y^2 = 56x^2 + 1 \tag{1}$$

with the least positive integer solution

$$x_0 = 2 \qquad y_0 = 15$$

whose general solution (x_n, y_n) is given by

$$x_n = \frac{1}{4\sqrt{14}} g_n \tag{2}$$

$$y_n = \frac{1}{2} f_n \tag{3}$$

in which

$$f_n = \left(15 + 4\sqrt{14}\right)^{n+1} + \left(15 - 4\sqrt{14}\right)^{n+1}$$

$$g_n = \left(15 + 4\sqrt{14}\right)^{n+1} - \left(15 - 4\sqrt{14}\right)^{n+1}$$

where $n = 0,1,2,\ldots$

It is to be noted that the equation (1) represents a hyperbola and the pair (x_n, y_n), $n = 0,1,2,\ldots$ gives the integral points on it.

A few numerical examples are presented in the table below:

n	x_n	y_n
0	2	15
1	60	449
2	1798	13455
3	53880	403201
4	1614602	12082575
5	48384180	362074049

A few interesting properties among the solutions are presented below:

1. For values of n given by $n = 0,1,2,\ldots$ the values of x_n are even numbers and y_n are odd numbers.
2. $x_{2n+1} \equiv 0 \pmod 3$, $n = 0,1,2,\ldots$
3. $y_{2n} \equiv 0 \pmod 5$, $n = 0,1,2,\ldots$

4. The recurrence relation satisfied by x_n and y_n are correspondingly exhibited below:

$$x_{n+2} - 30x_{n+1} + x_n = 0$$
$$y_{n+2} - 30y_{n+1} + y_n = 0$$

5. Each of the following expressions is equivalent to x_n.
   - $15x_{n+1} - 2y_{n+1}$
   - $30x_{n+1} - x_{n+2}$
   - $x_{n+2} - 4y_{n+1}$
   - $\frac{1}{16}[449x_{n+1} + 449x_{n+2} - 62y_{n+2}]$
   - $\frac{1}{112}[15y_{n+2} - 449y_{n+1}]$
   - $\frac{1}{113}[30x_{n+1} - x_{n+2} - 449y_{n+1} + 15y_{n+2}]$
6. Each of the following expressions is equivalent to y_n.
   - $15y_{n+1} - 112x_{n+1}$
   - $y_{n+2} - 224x_{n+1}$
   - $30y_{n+1} - y_{n+2}$
   - $449y_{n+2} - 3360x_{n+2}$
   - $\frac{1}{2}[15x_{n+2} - 449x_{n+1}]$
   - $\frac{1}{15}[449y_{n+1} - 112x_{n+2}]$
7. Each of the following expressions is a nasty number.
   - $12(112x_{2n} + 15y_{2n} + 1)$
   - $\frac{1}{5}[x_{2n+3} - 449x_{2n+1} + 60]$
   - $\frac{4}{5}[y_{2n+2} - 112x_{2n+1} + 15]$
   - $12(15y_{2n+2} - 112x_{2n+2} + 1)$
8. Each of the following expressions is a perfect square.
   - $2(y_{2n+3} - 224x_{2n+2} + 1)$
   - $2(30y_{2n+2} - y_{2n+3} + 1)$
   - $2(449y_{2n+3} - 3360x_{2n+3} + 1)$
   - $15x_{2n+3} - 449x_{2n+2} + 2$
9. Each of the following expressions is a cubic integer.
   - $15x_{3n+2} - x_{3n+1} + 180y_{n+1} - 6y_{n+2}$

- $60y_{3n+1}-2y_{3n}-672x_{n+1}+90y_{n+1}$
- $448x_{3n+1}+2y_{3n}+180y_{n+1}-6y_{n+2}$

REMARKABLE OBSERVATIONS

A. Let r and s be any two non-zero distinct positive integers and that $r>s$. Let (α,β,γ) be the Pythagorean triangle with r, s as its generators such that $\alpha=2rs,\beta=r^2-s^2,\gamma=r^2+s^2$.The different choices of r and s and the corresponding relation between the sides of the Pythagorean triangle are exhibited below.

S.No	r	s	Relation
1	x_n+y_n	x_n	$27\gamma+\alpha+1\equiv 0(\mathrm{mod}\ 28)$
2	x_n+y_n	y_n	$111\gamma+\beta+2\equiv 0(\mathrm{mod}\ 112)$
3	x_n+y_n	y_n	$56\gamma-56\alpha+1=t_{4,s}$
4	x_n+y_n	x_n	$\gamma-\alpha-1=56t_{4,s}$
5	$y_n+\frac{x_n-1}{2}$	$\frac{x_n-1}{2}$	$\gamma-\alpha-57=448t_{3,s}$
6	$x_n+\frac{y_n-1}{2}$	$\frac{y_n-1}{2}$	$7\gamma-7\alpha=t_{3,s}$
7	$y_n+\frac{x_n-1}{2}$	$\frac{x_n-1}{2}$	$\gamma-\alpha-1=56Ct_{8,s}$
8	$x_n+\frac{y_n-1}{2}$	$\frac{y_n-1}{2}$	$56\gamma-56\alpha+1=Ct_{8,s}$
9	$y_n+\frac{x_n+1}{3}$	$\frac{x_n+1}{3}$	$\gamma-\alpha-57=168t_{8,s}$
10	$x_n+\frac{y_n+1}{3}$	$\frac{y_n+1}{3}$	$56\gamma-56\alpha=3t_{8,s}$
11	$y_n+\frac{x_n-1}{2}$	$\frac{x_n-1}{2}$	$\gamma-\alpha+55=112Ct_{4,s}$
12	$x_n+\frac{y_n-1}{2}$	$\frac{y_n-1}{2}$	$28\gamma-28\alpha+1=Ct_{4,s}$
13	$y_n+\frac{x_n-5}{10}$	$\frac{x_n-5}{10}$	$\gamma-\alpha-841=560Ct_{20,s}$
14	$x_n+\frac{y_n-5}{10}$	$\frac{y_n-5}{10}$	$28\gamma-28\alpha-7=5Ct_{20,s}$
15	$y_n+\frac{x_n+3}{6}$	$\frac{x_n+3}{6}$	$\gamma-\alpha-169=336S_s$
16	$x_n+\frac{y_n+3}{6}$	$\frac{y_n+3}{6}$	$28\gamma-28\alpha-1=3S_s$

B. Employing the solutions (x,y) (1) the following expressions among the special Polygonal and Pyramidal numbers are given below.

1. $56\left[\dfrac{P_x^5}{t_{3,x}}\right]^2+1$ is a perfect square.

2. $\left[\dfrac{12P_{y-1}^6}{t_{6,2(y-1)}}\right]^2-56\left[\dfrac{2P_{x-1}^5}{t_{4,x-1}}\right]^2=1$

3. $\left[\dfrac{6P_{y-1}^4}{t_{3,2(y-1)}}\right]^2-2\left[\dfrac{6P_x^4}{t_{6,x+1}}\right]^2-1$ is a nasty number.

4. $\left[\dfrac{2P_{y-1}^8}{t_{6,y-1}}\right]^2-2\left[\dfrac{2P_{x-1}^5}{t_{4,x-1}}\right]^2-1$ is a nasty number.

5. $\dfrac{1}{56}\left\{\left[\dfrac{12P_y^3}{Ct_{4,y+1}-1}\right]^2-1\right\}$ is a perfect square.

CONCLUSION

One may search for other patterns of solutions and their corresponding properties.

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