International journal of Engineering Research-Online
A Peer Reviewed International Journal
Articles available online http://www.ijoer.in

Vol.1., Issue.3., 2013

RESEARCH ARTICLE

ISSN: 2321-7758

ON EQUAL SUMS OF LIKE POWERS $x^2 + y^2 + z^2 = w^2 + U^2$

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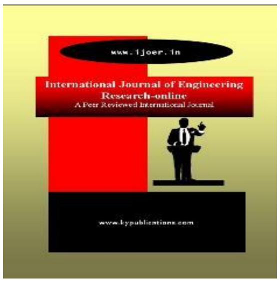
Article Received: 15/10/2013 | Article Revised on:30/11/2013 | Article Accepted on:01/12/2013



ABSTRACT

The quadratic Diophantine Equation with five unknowns given by $x^2 + y^2 + z^2 = w^2 + U^2$ is analyzed for its patterns of non – zero distinct integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

KEY WORDS: Quadratic equation, Integral solutions, Special polygonal numbers, Pyramidal numbers.

INTRODUCTION

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, homogeneous or non-homogeneous Quadratic Diophantine equation with two or more variables have been an interest to various mathematicians since antiquity [1-4]. In particular, one may refer [5-15] for different choices of Quadratic Diophantine equations with four unknowns. This communication concerns with an interesting homogeneous Quadratic Diophantine equation with five unknowns $x^2 + y^2 + z^2 = w^2 + U^2$ for obtaining its non-trivial integral solutions. A few interesting relations between the solutions and special polygonal numbers are presented.

NOTATIONS USED

- $t_{m,n}$ - Polygonal number of rank n with size m.
- P_n^m - Pyramidal number of rank n with size m.
- so_n - Stella octangular number of rank n
- s_n - Star number of rank n
- pr_n - Pronic number of rank n
- Pt_n - Pentatope number of rank n
- J_n - Jacobsthal number of rank n
- j_n - Jacobsthal – Lucas number of rank n

METHOD OF ANALYSIS

The quadratic Diophantine Equation with five unknown to be solved for its non-zero integral solution (x, y, z, w, U) is

$$x^2 + y^2 + z^2 = w^2 + U^2 \tag{1}$$

Which is satisfied by

$(2m, n, m+n, m-n, 2m+n)$, $(2m, 2n, 2n+m, 2n-m, 2n+2m)$

$(2m+1, 2n, 2m+n+1, 2m-n+1, 2m+2n+1)$, $(ad, bd, t, a+bd, ad-b)$ in which $a^2 + b^2 = t^2$

and $(ad+qr, bd-pr, t, a+bd-pr, ad-b+qr)$ where $r = \frac{a^2+b^2-t^2}{2}, ap+bq=1$

However, we have other patterns of solutions of (1) and they are illustrated below.

Pattern -1

The substitution $U = x + y$ (2)

in (1) leads to $z^2 - w^2 = 2xy$ (3)

Taking $x = 2^{2\alpha-1} y$ (4)

in (3) it is written as $z^2 - w^2 = (2^\alpha y)^2$ (5)

which is satisfied by

$$\begin{aligned} w &= p^2 - q^2 \\ y &= \frac{1}{2^\alpha}(2pq) \\ z &= p^2 + q^2 \end{aligned} \tag{6}$$

Write p as $p = 2^\alpha p, (\alpha > 0)$ (7)

From (4), (6), (7) and (2), the corresponding nonzero distinct integral solutions of (1) are given by

$$\begin{aligned} x &= 2^{2\alpha} Pq \\ y &= 2Pq \\ z &= 2^{2\alpha} P^2 + q^2 \\ w &= 2^{2\alpha} P^2 - q^2 \\ U &= 2Pq(2^{2\alpha-1} + 1) \end{aligned}$$

Properties:

1. $z(1,q) + w(1,q) + (-1)^{2\alpha+1} = j_{2\alpha+1}$
2. $U(t_{3,n}, t_{3,n+2}) - y(t_{3,n}, t_{3,n+2}) = 3.2^{2\alpha+3} Pt_n$
3. $x(p,q) + y(p,q) - U(p,q) + z(p,q) + w(p,q) \equiv 0 \pmod{2^{2\alpha+1}}$
4. $x(p, p+1) + y(p, p+1) + U(p, p+1) = [3J_{2\alpha+1} + (-1)^{2\alpha+1}]Pr_p + 8t_{3,p}$

5. $x(p,2p^2+1)+2y(p,2p^2+1)-U(p,2p^2+1)=6oh_p$

Each of the following represents a Nasty number

$3\{z(p,q)-w(p,q)\}$

$3\{x(p,q)+y(p,q)-U(p,q)+z(p,q)+w(p,q)\}$

Note that (5) is also satisfied by

$$w=2pq$$
$$y=\frac{1}{2^{\alpha}}(p^2-q^2)$$
$$z=p^2+q^2$$

For this choice , the corresponding nonzero distinct integral solutions of (1) are given by

$$x=2^{4\alpha-1}(P^2-Q^2)$$
$$y=2^{\alpha}(P^2-Q^2)$$
$$z=2^{2\alpha}(P^2+Q^2)$$
$$w=2^{2\alpha+1}PQ$$
$$U=(2^{4\alpha-1}+2^{\alpha})(P^2-Q^2)$$

Properties:

1. $(w,2^{\alpha},z)$ forms a Pythagorean triangle.
2. $z(P,Q)+w(P,Q)=2^{2\alpha}t_{4,P+Q}$
3. $x(P,Q)+y(P,Q)-U(P,Q)+w(P,P(P+1))=2^{(\alpha+1)^2}P_P^5$
4. $x(P,Q)+y(P,Q)-U(P,Q)+z(P,Q)+t_{4,2^{\alpha}Q}\equiv 0(\mathrm{mod}\, 2^{2\alpha})$

Each of the following represents a Nasty number

$x(P,P)+y(P,P)+3z(P,P)$

$6(z(P,Q)-w(P,Q)\}$

Pattern-2:

The substitution of the linear transformations

$$x=u+v.y=u-v, z=2u, w=u+r, U=u-r, (u\neq v\neq r) \quad (8)$$

in (1) leads to $v^2+2u^2=r^2$

which is satisfied by

$$r=p^2+2q^2$$
$$u=2pq \qquad p>q>0 \quad (9)$$
$$v=p^2-2q^2$$

In view of (8) and (9), the corresponding nonzero distinct integral solutions of (1) are given by

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$$x = p^2 - 2q^2 + 2pq$$
$$y = -p^2 + 2q^2 + 2pq$$
$$z = 2pq$$
$$w = p^2 + 2q^2 + 2pq$$
$$U = -p^2 - 2q^2 + 2pq$$

Properties:

1. $x(t_{3,n}, t_{3,n+2}) + y(t_{3,n}, t_{3,n+2}) = 96Pt_n = z(t_{3,n}, t_{3,n+2})$
2. $x(t_{3,p}, p) + y(t_{3,p}, p) + w(t_{3,p}, p) + U(t_{3,p}, p) = 8P_p^5$
3. $x(p,q) + y(p,q) = z(p,q) = w(p,q) + U(p,q)$
4. $x(p,2p^2-1) + y(p,2p^2-1) + z(p,2p^2-1) - w(p,2p^2-1) - U(p,2p^2-1) = 4so_p$

Each of the following represents a Nasty number

$$3\{x(p,(p+1)(p+2)) + w(p,(p+1)(p+2))\} - 72P_p^3$$
$$2\{x(p,p) + y(p,p) + U(p,p) + z(p,p) + w(p,p)\}$$

REMARKABLE OBSERVATIONS:

I: If the non-zero integer quintuple $(x_0, y_0, z_0, w_0, U_0)$ is any given solution of (1) then each of the following three quintuple (I to III) expressed in matrix forms also satisfies (1)

Quintuple I :

$$\begin{pmatrix} x_n \\ y_n \\ z_n \\ w_n \\ U_n \end{pmatrix} = \begin{pmatrix} a_{12}^{n-1} + a_{13}^{n-1} + a_{14}^{n-1} & a_{11}^{n} - 1 & a_{11}^{n} - 1 & a_{14}^{n-1} + a_{41}^{n-1} + a_{44}^{n-1} \\ a_{11}^{n} - 1 & a_{12}^{n-1} + a_{13}^{n-1} + a_{14}^{n-1} & a_{11}^{n} - 1 & a_{14}^{n-1} + a_{41}^{n-1} + a_{44}^{n-1} \\ a_{11}^{n} - 1 & a_{11}^{n} - 1 & a_{12}^{n-1} + a_{13}^{n-1} + a_{14}^{n-1} & a_{14}^{n-1} + a_{41}^{n-1} + a_{44}^{n-1} \\ a_{14}^{n-1} + a_{41}^{n-1} + a_{44}^{n-1} & a_{14}^{n-1} + a_{41}^{n-1} + a_{44}^{n-1} & a_{14}^{n-1} + a_{41}^{n-1} + a_{44}^{n-1} & a_{34}^{n} + a_{43}^{n} + a_{41}^{n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \\ U_0 \end{pmatrix}$$

in which
$$\begin{pmatrix} a_{11}^0 & a_{12}^0 & a_{13}^0 & a_{14}^0 & a_{15}^0 \\ a_{21}^0 & a_{22}^0 & a_{23}^0 & a_{24}^0 & a_{25}^0 \\ a_{31}^0 & a_{32}^0 & a_{33}^0 & a_{11}^0 & a_{35}^0 \\ a_{41}^0 & a_{42}^0 & a_{43}^0 & a_{44}^0 & a_{45}^0 \\ a_{51}^0 & a_{52}^0 & a_{53}^0 & a_{54}^0 & a_{55}^0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Quintuple II :

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \\ U_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 00 \\ -1 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \\ U_0 \end{pmatrix}$$

Quintuple III :

$$\begin{pmatrix} x_n \\ y_n \\ z_n \\ w_n \\ U^*{}_n \end{pmatrix} = 3^{2n-2} \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 2 & 2 \\ 0 & -4 & -5 & 4 & 4 \\ 0 & -2 & -4 & 5 & 2 \\ 0 & -2 & -4 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \\ U_0 \end{pmatrix}$$

CONCLUSION:

It is worth mentioning here that,by employing the method of factorization and applying the method of cross-multiplication, one may obtain different values for y,z and w satisfy (5). To conclude, one may search for other patterns of solutions and their corresponding properties.

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