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EXPONENTIAL DIOPHANTINE EQUATION IN THREE VARIABLES $7^x + 7^{2y} = z^2$

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ABSTRACT

In this paper, we find solutions of exponential Diophantine equation $7^x + 7^{2y} = z^2$ in non- negative integers x, y and z. Two different approaches are illustrated for finding the non – negative integer points satisfying the exponential Diophantine equation under consideration.

Keywords— Exponential Diophantine equation, Congruence, Integral solutions, Catalan's conjecture.

I. INTRODUCTION

Number Theory is a branch of pure mathematics devoted primarily to the study of integers. Diophantine Analysis deals with various techniques of solving Diophantine equations in multivariables and multi degrees. If a Diophantine equation has variables occurring as exponents, it is an exponential Diophantine equation. For example, the $Ramanujan - Nagell$ equation $2^x - 7 = x^2$ and the equation of the $Fermat - Catalan$ conjecture $a^m + b^m = c^k$.

In this paper, two different exponential Diophantine equations are considered for finding non-trivial integral solutions.

II. PRELIMINARIES

In this section we use the factorisable method and Catalan's conjecture to prove the two lemmas.

A. PROPOSITION 2.1: $(3,2,2,3)$ IS A UNIQUE SOLUTION (A, B, X, Y) FOR THE DIOPHANITNE EQUATION $A^X - B^Y = 1$ WHERE A, B, X AND Y ARE INTEGERS WITH $\mathrm{MIN}\{A, B, X, Y\} > 1$.

B. LEMMA 2.2: THE DIOPHANTINE EQUATION $7^X + 1 = Y^2$, WHERE X AND Y ARE NON-NEGATIVE INTEGERS, HAS NO NON- NEGATIVE INTEGER SOLUTIONS.

Proof: Let x and y be non-negative integers such that $7^x + 1 = y^2$. If $x = 0$ then $y^2 = 2$, which is not possible in integers. If $y = 0$, then $7^x = -1$, which is also impossible. Now for $x, y > 0$, Consider, $7^x = y^2 - 1 = (y+1)(y-1)$. Let $y + 1 = 7^q, y - 1 = 7^p$, where $p < q$ and $p + q = x$.

Then $7^p(7^{q-p} - 1) = 2$. Thus, if $7^p = 1 \Rightarrow p = 0$ and $7^{q-p} = 3$, which is also impossible.

For the choice $7^p = 2$ and $7^{q-p} - 1 = 1$, there exist no integer solution. Hence $7^x + 1 = y^2$ has no solution in non-negative integers.

Another Proof: Suppose that there are non-negative integers x and y such that $7^x + 1 = y^2$. If $x = 0$, then $y^2 = 2$ which is impossible. Then $x \geq 1$. Thus, $y^2 = 7^x - 1 \geq 7 + 1 \geq 8$. Then $y \geq 3$. Now, we consider on the equation $y^2 - 7^x = 1$. By proposition(2.1), we have $x = 1$. Then $y^2 = 8$. This is a contradiction. Hence, the equation $7^x + 1 = y^2$ has no non- negative integers.

C. LEMMA 2.3: The Diophantine equation$7^{2x} + 1 = y^2$, where x and y are non-negative integers, has no non- negative integer solutions.

Proof: Let x and y be non-negative integers satisfying $(7^2)^x + 1 = y^2$.

If $x = 0 \Rightarrow y^2 = 2$ which is not solvable in non-negative integers.

If $y = 0, (7^2)^x = -1$, which is also impossible. So let $x, y > 0$.

Consider $(7^2)^x + 1 = y^2 \Rightarrow y^2 - (7^x)^2 = 1$, which is not possible since the difference of two square can never be equal to one.

Hence the equation $7^{2x} + 1 = y^2$ has no solution in non-negative integers.

Another Proof: Suppose that there are non-negative integers x and y such that $49^x + 1 = y^2$.

If $x = 0$, then $y^2 = 2$ which is impossible. Then $x \geq 1$. Thus, $y^2 = 49^x + 1 \geq 50$. Then $y \geq 8$.

Now, we consider on the equation $y^2 - 49^x = 1$, By Proposition (2.1), we have $x = 1$. Then $y^2 = 50$. This is a contradiction. Hence the equation $49^x + 1 = y^2$ has no non-negative integer solution.

III. MAIN RESULTS

A. THEOREM 3.1: THE DIOPHANTINE EQUATION $7^x + (7^2)^y = z^2$ HAS NO NON-NEGATIVE INTEGER SOLUTION.

Proof: If $x = 0$, $(7^2)^y + 1 = z^2$ has no non-negative integer solution by lemma (2.3).

If $y = 0$, $7^x + 1 = z^2$ has no non – negative integer solution by lemma (2.2). If $z = 0, 7^x + (7^2)^y = 0$ which is not impossible for non-negative integers x and y. So let $x, y, z > 0$.

Consider $7^x = z^2 - (7^2)^y = (z + 7^y)(z - 7^y) = (7^q)(7^p)$, where $p < q$ and $p + q = x$.

Then $7^p(7^{q-p} - 1) = 2(7^y)$. Thus, if $7^p = 7^y \Rightarrow p = y$ and $7^{q-p} = 3$, which is also impossible.

For the choices $7^p = 2$ and $7^{q-p} - 1 = 7^y$, there exist no integer solution.

Hence $7^x + (7^2)^y = z^2$ has no solution in non-negative integers.

Another Proof: Let x, y and z be non- negative integers such that $7^x + 7^{2y} = z^2$. By lemma (2.2)and (2.3) we have $x, y \geq 1$. Note that z is even. It follows that $z^2 \equiv 0(mod\ 3)$ or $z^2 \equiv 1(mod\ 3)$. Note that $7^x \equiv 1(mod\ 3)$ and $(7^2)^y \equiv 1(mod\ 3)$ .Thus $z^2 \equiv 2(mod\ 3)$. This is impossible. Hence, the equation $7^x + 49^y = z^2$ has no non- negative integer solution.

IV. CONCLUSIONS

Our goal is to examine various exponential Diophantine equation in three variables for finding integer solution. This paper outlines two different methods for solving $7^x + (7^2)^y = z^2$.

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