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ON TERNARY QUADRATIC DIOPHANTINE EQUATION: $2x^2 + 2y^2 - 3xy = 2z^2$

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ABSTRACT

The Ternary Quadratic Diophantine Equation given by $2x^2 + 2y^2 - 3xy = 2z^2$ is analysed for its patterns of non-zero distinct integral solutions.
A few interesting relations between the solutions and special polygonal numbers are exhibited.

KEY WORDS: Ternary Quadratic, Integral solutions
MATHEMATICS SUBJECT CLASSIFICATION:11D09

INTRODUCTION

The Ternary Quadratic Diophantine Equation offers an unlimited field for research because of their variety[1-5],For an extensive review of various problems, One may refer[6-21].This communication concerns with yet another interesting Ternary Quadratic Diophantine Equation $2x^2 + 2y^2 - 3xy = 2z^2$ for determining its infinitely many non-zero integral solutions. Also, a few interesting relations among the solutions have been presented.

NOTATIONS USED:

$T_{m,n}$ = Polygonal Number of rank n with sides m

p_n^m = Pyramidal number of rank n with sides m

j_n =Jacobsthal-Lucas number of rank n

J_n = Jacobsthal number of rank n

METHOD OF ANALYSIS: The ternary quadratic equation to be solved for its non-zero distinct integral solutions is,

$2x^2 + 2y^2 - 3xy = 2z^2$ ……………………..(1)

To start with, it is seen that(1) is satisfied by the following triples of integers: (x, y, z) = (538,112,318), ($700k^2$ $+160k+4,120k+6,700k^2+70k+4$),($100k^2+160k+28,120k+42,100k^2+70k+28$), ($20k^2+160k+140,120k+210,20k^2+70k+140$),($700p^2+4q^2+160pq,6q^2+120pq,700p^2+4q^2+70pq$). However ,we have other patterns of solutions which are illustrated below:

The substitution of linear transformations

$x = u + v \;, y = u - v \;, (u \neq v \neq 0)$-----(2)

in (1) leads to $u^2 + 7v^2 = 2z^2$------------------(3)

Different patterns of solutions of (1) are presented below.

Pattern: I

Write 2 as, $2 = \frac{(1+i\sqrt{7})(1-i\sqrt{7})}{4}$-------------------(4)

Assume $z = z(a,b) = a^2 + 7b^2$---------------(5)

where a, b are non-zero distinct integers

Substituting $(4),(5) in (3)$,it is written in the factorizable form as

$$(u+i\sqrt{7}v)(u-i\sqrt{7}v) = \frac{(1+i\sqrt{7})(1-i\sqrt{7})}{4}(a+i\sqrt{7}b)^2(a-i\sqrt{7}b)^2$$

Equating the real and imaginary parts, we get

$$u = \frac{1}{2}(a^2 - 7b^2 - 14ab)$$

$$v = \frac{1}{2}(a^2 - 7b^2 + 2ab)$$

Substituting u and v values in (2), we get

$x = x(a,b) = a^2 - 7b^2 - 6ab$---------(6)

$y = y(a,b) = -8ab$--------------(7)

Thus (6) ,(7) and (5) represent non-zero distinct integral solutions of (1) in two parameters.

A few interesting properties observed as follows.

Properties:

(i) $x(a,1) - 2t_{3,a} \equiv 0(mod\ 7)$

(ii) $x(a,1) + z(a,1) - 4t_{3,a} \equiv 0(mod\ 4)$

(iii) $2z(a+3,a+3) - 32t_{3,a} \equiv 0(mod\ 2)$

(iv) $2z(a+1,a+1) - 4t_{3,a} - 28t_{3,a} \equiv 0(mod\ 2)$

Remark: Note that, in addition to (4), one may write 2 in the following two forms:

Form:I

$$2 = \frac{(5+i\sqrt{7})(5-i\sqrt{7})}{16}$$

The integer solutions obtained for$(1)$are given by

$$x = x(a,b) = 6a^2 - 42b^2 - 4ab$$

$$y = y(a,b) = 4a^2 - 28b^2 - 246ab$$

$$z = z(a,b) = 4a^2 + 28b^2$$

Properties:

(i) $x(a,1) + y(a,1)) - t_{14,a} - t_{10,a} \equiv 0(mod\ 10)$

(ii) $x(a,1) + z(a,1) - t_{22,a} \equiv -4(mod\ 5)$

(iii) $x(2^n,1) = 6(j_{2n} - 1) - 4(j_n - (-1)^n) - 42$

(iv) $y(2^n,1) = 4(j_{2n} - 1) - 24(j_n - (-1)^n) - 28$

Form:II

$$2 = \frac{(11+i\sqrt{7})(11-i\sqrt{7})}{64}$$

The corresponding integer solutions obtained are

$x = x(a,b) = 6a^2 - 42b^2 + 4ab$

$y = y(a,b) = 5a^2 - 35b^2 - 18ab$

$z = z(a,b) = 4a^2 + 28b^2$

Properties:

(i) $x(a,1) - t_{14,a} \equiv 0(mod\ 3)$

(ii) $x(2^n,1) = 2[3(j_{2n} - 1) - 2^{n+1} - 21]$

(iii) $y(2^n,1) = 5(j_{2n} - 1) - 18(j_n - (-1)^n) - 35$

(iv) $z(2^n,1) = 4(j_n - (-1)^n) + 28$

Pattern:II

Treating (1) as a quadratic in x and solving for x we get

$x = \frac{1}{4}\left(3y \pm \sqrt{(16z^2 - 7y^2)}\right)$-----------------(7)

let

$\alpha^2 = 16z^2 - 7y^2$

which is written as

$\alpha^2 - (3z)^2 = 7(z^2 - y^2)$---------------------- (8)

Express the above equation in the form of ratio as

$\frac{\alpha+3z}{z+y} = \frac{7(z-y)}{\alpha-3z} = \frac{A}{B}$, $B \neq 0$-----------------(9)

which is equivalent to the following two equations

$B\alpha + 3\alpha z - Az - Ay = 0$

$B\alpha + z(3 - A) - Ay = 0$

Applying the method of cross multiplication, we have

$$\left.\begin{aligned} y &= A^2 - 7B^2 - 6AB \\ z &= -(A^2 + 7B^2) \end{aligned}\right\}\text{---------------------(10)}$$

Substituting the values of y and z in (7) and taking the positive sign on the RHS of (7),

we have

$x = -8AB$---------------------(11)

Thus (10) and (11) represent the integer solutions of (1)

Also, taking the negative sign on the RHS of (7),the corresponding integer solutions are

$$x = 6A^2 - 42B^2 - 4AB$$

$$y = 4A^2 - 28B^2 - 24AB$$

$$z = -(4A^2 + 28B^2)$$

APPLICATIONS:

I. Employing the solutions (x, y, z) of (1) each of following expressions among the special polygonal &pyramidal numbers is a perfect square.

1. $2\left(\frac{p_x^5}{t_{3,x}}\right)^2 + 2\left(\frac{2p_{y-1}^5}{t_{4,y-1}}\right)^2 + 3\left(\frac{p_x^5}{t_{3,x}}\right)^2 * \left(\frac{2p_{y-1}^5}{t_{4,y-1}}\right)^2$

2. $2\left(\frac{p_x^5}{t_{3,x}}\right)^2 + 2\left(\frac{3p_{y-2}^8}{t_{8,y-2}}\right)^2 + 3\left(\frac{p_x^5}{t_{3,x}}\right)^2 * \left(\frac{3p_{y-2}^8}{t_{8,y-2}}\right)^2$

II. If x, y are taken as the generators of a Pythagorean triangle, than twice the Hypotenuse is congruent to thrice the product of its generators under module 2.

III. Consider x and y to be the length and breadth of a rectangle R,whose

Area = A

Perimeter = P

Length of the diagonal = L

Then ,it is noted that

1. $3(2L^2 - 3A)$ *is a nasty number*
2. $P^2 - 14A \equiv 0(mod\ 4)$
3. $P^2 = 4L^2 + 8A$

CONCLUSION: To conclude, one may search for other patterns of solutions and their corresponding properties. Also, as the quadratic equations are richin variety, one may attempt to determine integer solutions to other choices of quadratic equations with variables ≥ 3.

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