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FLEXURAL ANALYSIS OF CANTILEVER BEAM WITH CUBIC LOAD USING REFINED SHEAR DEFORMATION THEORY

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ABSTRACT

Taking into account transverse shear deformation effects, the flexural analysis of isotropic deep beams using fifth order shear deformation theory is presented. The number of variables in the present theory is same as that in the first order shear deformation theory. The function is used in displacement field is in terms of thickness coordinate to represent the shear deformation effects. The transverse shear stresses can be obtained directly from the use of constitutive relations with accuracy, satisfying the shear stress free conditions on the top and bottom surfaces of the beam. Hence, the theory obviates the need of shear correction factor. Governing differential equations and boundary conditions are obtained by using the principle of virtual work. The deep isotropic cantilever beam is considered for the numerical studies to show the efficiency of the theory. It has been shown that the theory is capable of predicting the local effects due to cubic load. The results obtained for flexure for mentioned beam using the fifth order theory are presented and discussed with those of other theories, and are found to agree well with the exact elasticity results.

I. Introduction

As the beams and plates are the most interesting areas of research, and these component is being regularly used in daily engineering applications of Engineering. In 1705 Euler and Bernoulli gave the Classical Beam Theory on the first mathematical model of nature of the resistance of beam developed in 1638 by Galileo. Saint Venant (1856) presented the complete solution of the beam problems considering bending and shear stresses. After the Krichhoff (1850) these theories get matured.

It is well-known that elementary theory of bending of beam based on Euler-Bernoulli hypothesis disregards the effects of the shear deformation and stress concentration. The theory is suitable for slender beams and is not suitable for thick or deep beams since it is based on the assumption that the transverse normal to neutral axis remains so during bending and after bending, implying that the transverse shear strain is zero. Since theory neglects the transverse shear deformation. It underestimates deflections in case of thick beams where shear deformation effects are significant.

Bresse [1], Rayleigh [2] and Timoshenko [3] were the pioneer investigators to include refined effects such as rotatory inertia and shear deformation in the beam theory. Timoshenko showed that the effect of transverse vibration of prismatic bars. This theory is now widely referred to as Timoshenko beam theory or first order shear deformation theory (FSDT) in the literature. In this theory transverse shear strain distribution is assumed to be constant through the beam thickness and thus requires shear correction factor to appropriately represent the strain energy of deformation.

Cowper [4] has given refined expression for the shear correction factor for different cross-sections of beam. The accuracy of Timoshenko beam theory for transverse vibrations of simply supported beam in respect of the fundamental frequency is verified by Cowper [5] with a plane stress exact elasticity solution.

To remove the discrepancies in classical and first order shear deformation theories, higher order or refined shear deformation theories were developed and available in the open literature for static and vibration analysis of beam

Levinson [6], Bickford [7] Rehfield and Murty [8], Krishna Murthy [9], Baluch et al. [10], Bhimaraddi and Chandrashekhara [11] presented parabolic shear deformation theories assuming a higher variation of axial displacement in terms of thickness coordinate. These theories satisfy shear stress free boundary conditions on top and bottom surfaces of beam and thus obviate the need of shear correction factor.

Irretier [12] studied the refined dynamical effects in linear, homogenous beam according to theories, which exceed the limits of the Euler-Bernoulli beam theory. These effects are rotary inertia, shear deformation, rotary inertia and shear deformation, axial pre-stress, twist and coupling between bending and torsion. Kant and Gupta [13], and Heyliger and Reddy [14] presented finite element models based on higher order shear deformation uniform rectangular beams. However, these displacement based finite element models are not free from phenomenon of shear locking [15,16].

There is another class of refined theories, which includes trigonometric functions to represent the shear deformation effects through the thickness. Vlasov and letont'ev [17] and Stein [18] developed refined shear deformation theories for thick beams including sinusoidal function in terms of thickness coordinate in displacement field. However, with these theories shear stress free boundary conditions are not satisfied at top and bottom surfaces of the beam. This discrepancy is removed by Rao [19] in a refined theory developed for beam. However, formulation of theory is variationally inconsistent.

A study of literature by Ghugal and Shimpi [20] indicates that the research work dealing with flexural analysis of thick beams using refined trigonometric and hyperbolic shear deformation theories is very scarce and is still in infancy.

I. Development of Theory

The beam under consideration as shown in Fig.1 occupies in Cartesian coordinate system the region

$$0 \le x \le L \ ; \quad -\frac{b}{2} \le y \le \frac{b}{2} \ ; \quad -\frac{h}{2} \le z \le \frac{h}{2}$$

where *x, y, z* are Cartesian coordinates, L and b are the length and width of beam in the x and y directions respectively, and h is the depth of the beam in the z-direction. The beam is made up of homogeneous, linearly elastic isotropic material.

A. *The displacement field*

The displacement field of the present beam theory is of the form [18] as given below:

$$u(x,z) = -z\frac{dw}{dx} + z\left[2 - \frac{4}{3}\left(\frac{z}{h}\right)^2 - \frac{16}{5}\frac{z^4}{h^4}\right]\phi(x)$$
$$w(x,z) = w(x) \qquad (1)$$

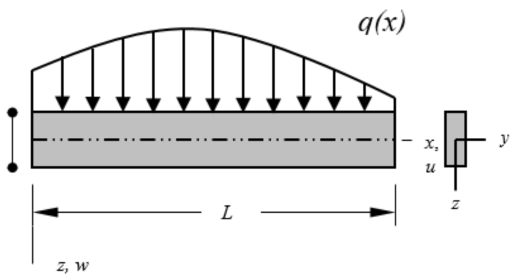

Fig. 1: Beam under bending in x-z plane

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where *u* is the axial displacement in x direction and *w* is the transverse displacement in z direction of the beam. The function is used in displacement field is in terms of thickness coordinate to represent the shear deformation effects. The represents rotation of the beam at neutral axis, which is an unknown function to be determined.

B. *Normal and Shear Strain*

$$\varepsilon_x = \frac{\partial u}{\partial z} = -z\frac{d^2 w}{dx^2} + z\left[2 - \frac{4}{3}\left(\frac{z}{h}\right)^2 - \frac{16}{5}\frac{z^4}{h^4}\right]\frac{d\phi}{dx} \quad (2)$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{dw}{dx} = \left[2 - 4\left(\frac{z}{h}\right)^2 - 16\frac{z^4}{h^4}\right]\phi(x) \quad (3)$$

C. *Stress Strain Relationship*

$$\sigma_x = E\varepsilon_x \qquad \tau_{zx} = G\gamma_{zx} \quad (4)$$

D. *Governing Equations and Boundary Conditions*

Using the expressions for strains and stresses (2) through (4) and using the principle of virtual work, variationally consistent governing differential equations and boundary conditions for the beam under consideration can be obtained. The principle of virtual work when applied to the beam leads to:

$$b\int_{x=0}^{x=L}\int_{z=-h/2}^{z=+h/2}\left(\sigma_x\delta\varepsilon_x + \tau_{zx}\delta\gamma_{zx}\right)dx\,dz - \int_{x=0}^{x=L} q(x)\,\delta w\,dx = 0 \quad (5)$$

where the symbol δ denotes the variational operator. Employing Green's theorem in Eqn. (4) successively, we obtain the coupled Euler-Lagrange equations which are the governing differential equations and associated boundary conditions of the beam. The governing differential equations obtained are as follows:

$$EI\frac{d^4 w}{dx^4} - \frac{12}{7}EI\frac{d^3\phi}{dx^3} - q(x) = 0 \quad (6)$$

$$\frac{12}{7}EI\frac{d^3 w}{dx^3} - 2.96EI\frac{d^2\phi}{dx^2} + 2.4635GA\phi(x) \quad (7)$$

The associated consistent natural boundary conditions obtained are of following form:

At the ends x = 0 and x = L

$$EI\frac{d^2 w}{dx^2} = EI\frac{d\phi}{dx} = w = 0 \quad (8)$$

Thus the boundary value problem of the beam bending is given by the above variationally consistent governing differential equations and boundary conditions.

The General Solution of Governing Equilibrium Equations of the Beam

The general solution for transverse displacement w(x) and warping function (x) is obtained using Eqns. (6) and (7) using method of solution of linear differential equations with constant coefficients. Integrating and rearranging the first governing Eqn. (6), we obtain the following equation

$$\frac{d^3 w}{dx^3} = A_0\frac{d^2\phi}{dx^2} + \frac{Q(x)}{EI} \quad (9)$$

where Q(x) is the generalized shear force for beam and it is given by $Q(x) = \int_0^x q\,dx + C_1$.

Now second governing Eqn (7) is rearranged in the following form:

$$\frac{d^3 w}{dx^3} = \frac{B_0}{A_0}\frac{d^2\phi}{dx^2} - \beta\phi \quad (10)$$

A single equation in terms of ϕ is now obtained using Eqns (9) and (10) as:

$$\frac{d^2\phi}{dx^2} - \lambda^2\phi = \frac{Q(x)}{\alpha EI} \quad (11)$$

where constants α , β and λ in Eqns. (10) and (11) are as follows

$$\alpha = \left(\frac{B_0}{A_0} - A_0\right),\quad \beta = \left(\frac{C_0}{A_0}\frac{GA}{EI}\right) \text{and } \lambda^2 = \frac{\beta}{\alpha}$$

The general solution of Eqn. (11) is as follows:

$$\phi(x) = C_2\cosh\lambda x + C_3\sinh\lambda x - \frac{Q(x)}{\beta EI} \quad (12)$$

The equation of transverse displacement *w(x)* is obtained by substituting the expression of ϕ (x) in Eqn. (12) and then integrating it thrice with respect to x. The general solution for w(x) is obtained as follows:

$$w(x)=\frac{1}{EI}\iiiint q\,dx\,dx\,dx\,dx+\begin{pmatrix}\frac{C_1x^3}{6}+\left(\frac{A_0}{B_0}\lambda^2-\beta\right)\frac{EI}{\lambda^3}\left(C_2\sinh\lambda x-C_3\cosh\lambda x\right)\\+C_4\frac{x^2}{2}+C_5x+C_6\end{pmatrix} \quad (13)$$

where C_1, C_2, C_3, C_4, C_5 and C_6 are arbitrary constants and can be obtained by imposing natural boundary / end conditions of beam.

II. Illustrative Example

In order to prove the efficacy of the present theory, the following numerical examples are considered. The following material properties for beam are used

E = 210 GPa, μ = 0.3 and ρ = 7800 Kg/m3

where *E* is the Young's modulus, ρ is the density, and *μ* is the Poisson's ratio of beam material.

Example

The cantilever beam is as shown in Fig. 2 subjected to varying load, $q(x)=q_0\left(\frac{x^3}{L^3}\right)$ on surface z = -h/2 acting in the downward z direction

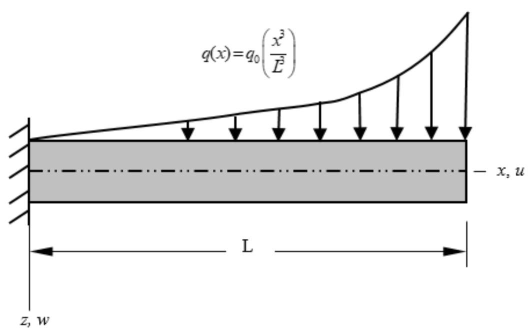

Fig. 2: Cantilever Beam with Cubic load

The final expressions for above loading & boundary conditions are obtained as follows:

1) Transverse displacement w(x) and ϕ (x)

$$w(x)=\frac{q_0L^4}{10Ebh^3}\begin{bmatrix}\left(\frac{x^7}{L^7}-5\frac{x^3}{L^3}+12\frac{x^2}{L^2}\right)-\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{x^5}{2L^5}-5\frac{x^2}{L^2}\right)\\+\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\frac{5}{2}\left(\frac{\cosh\lambda x-\sinh\lambda x-1}{\lambda L}+\frac{x}{L}\right)\end{bmatrix} \quad (14)$$

$$\phi(x)=\frac{q_0L}{4\beta EI}\left(\sinh\lambda x-\cosh\lambda x-\frac{x^4}{h^4}+1\right) \quad (15)$$

Substituting expressions for *w* and ϕ given by Eqns. (14) and (15) into Eqns. (1) through (4) the final expressions for axial displacement u, transverse displacement w, axial stresses σ_x and transverse shear stress τ_{zx} can be obtained respectively.

2) Expression for Axial Displacement ($\bar{u}$):

$$\bar{u}=\begin{bmatrix}-\frac{z}{h}\frac{L^3}{h^3}\frac{1}{10}\begin{bmatrix}\left(\frac{x^6}{L^6}-15\frac{x^2}{L^2}+24\frac{x}{L}\right)-\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(\frac{5x^4}{2L^4}-10\frac{x}{L}\right)\\+\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\frac{5}{2}\left(\sinh\lambda x-\cosh\lambda x+1\right)\end{bmatrix}\\+\frac{z}{h}\frac{A_0}{C_0}\frac{E}{G}\frac{L}{h}\frac{1}{4}\left[2-\frac{4}{3}\frac{z^2}{h^2}-\frac{16}{5}\frac{z^4}{h^4}\right]\left[\sinh\lambda x-\cosh\lambda x+1-\frac{x^4}{L^4}\right]\end{bmatrix} \quad (16)$$

3) Expression for Axial Stress ($\overline{\sigma_x}$):

$$\overline{\sigma_x}=\begin{bmatrix}-\frac{z}{h}\frac{L^2}{h^2}\frac{1}{10}\begin{bmatrix}\left(6\frac{x^5}{L^5}-30\frac{x}{L}+24\right)-\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(10\frac{x^3}{L^3}-10\right)\\+\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\frac{5}{2}\left(L\lambda\cosh\lambda x-L\lambda\sinh\lambda x\right)\end{bmatrix}\\+\frac{z}{h}\frac{A_0}{C_0}\frac{E}{G}\frac{1}{4}\left[2-\frac{4}{3}\frac{z^2}{h^2}-\frac{16}{5}\frac{z^4}{h^4}\right]\left[L\lambda\cosh\lambda x-L\lambda\sinh\lambda x-4\frac{x^3}{L^3}\right]\end{bmatrix} \quad (17)$$

4) Expression for Transverse shear Stress using equilibrium equation ($\overline{\tau_{ZX}^{EE}}$):

It is obtained using stress equilibrium equation of two dimensional elasticity which is as follows:

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{zx}}{\partial z}=0 \quad (18)$$

$$\overline{\tau_{zx}^{EE}}=\begin{bmatrix}-\frac{1}{80}\frac{L}{h}\left[1-4\frac{z^2}{h^2}\right]\begin{bmatrix}\left(30\frac{x^4}{L^4}-30\right)-\frac{B_0}{C_0}\frac{E}{G}\frac{h^2}{L^2}\left(30\frac{x^2}{L^2}\right)\\+\frac{A_0^2}{C_0}\frac{E}{G}\frac{h^2}{L^2}\frac{5}{2}\left(L^2\lambda^2\sinh\lambda x-L^2\lambda^2\cosh\lambda x\right)\end{bmatrix}\\-\frac{A_0}{C_0}\frac{E}{G}\frac{h}{L}\frac{1}{4}\left[\frac{z^2}{h^2}-\frac{1}{3}\frac{z^4}{h^4}-\frac{8}{15}\frac{z^6}{h^6}-\frac{53}{240}\right]\left[L^2\lambda^2\sinh\lambda x-L^2\lambda^2\cosh\lambda x-12\frac{x^2}{L^2}\right]\end{bmatrix} \quad (19)$$

5) Expression for transverse shear stress obtained from constitutive relationship ($\overline{\tau_{ZX}^{CR}}$):

$$\overline{\tau_{zx}^{CR}}=\frac{1}{4}\frac{L}{h}\frac{A_0}{C_0}\left[2-4\frac{z^2}{h^2}-16\frac{z^4}{h^4}\right]\left[\sinh\lambda x-\cosh\lambda x+1-\frac{x^4}{L^4}\right] \quad (20)$$

III. Results

In this paper, the results for in plane displacement, transverse displacement, axial and transverse stresses are presented in the following non dimensional form for the purpose of presenting the results in this work.

For beam subjected to cubic load, q(x)

$$\overline{u}=\frac{Ebu}{qh},\overline{w}=\frac{10Ebh^3w}{qL^4},\overline{\sigma}_x=\frac{b\sigma_x}{q},\overline{\tau}_{zx}=\frac{b\tau_{zx}}{q}$$

Table 1: Non-Dimensional Axial Displacement ($\overline{u}$) At (X=0.25l, Z=H/2), Transverse Displacement ($\overline{w}$) At (X=0.25l, Z=0) Axial Stress ($\overline{\sigma_x}$) At (X=0.25l, Z=H/2) Maximum Transverse Shear Stress ($\overline{\tau_{zx}}$) (X=0, Z=0) Of The Beam For Aspect Ratio 4 & 10

Model	S	$\overline{w}$	$\overline{u}$	$\overline{\sigma}$	$\overline{\tau}_{zx}^{CR}$	$\overline{\tau}_{zx}^{EE}$
ETB	4	0.67188	−16.20078	−13.20469	NA	1.49414
FSDT		0.67179	−16.20703	−13.20469	1.17692	1.49414
HSDT		0.85533	−18.03826	−14.77891	1.50000	1.49360
TSDT		0.85523	−18.05677	−14.78062	1.54807	1.49355
V Order		0.85396	−17.98587	−14.76302	1.39175	1.49372
ETB	10	0.67188	−253.13721	−82.52930	NA	3.73535
FSDT		0.67187	−253.23486	−82.52930	2.94231	3.73535
HSDT		0.70123	−257.73092	−84.10352	3.75000	3.73513
TSDT		0.70122	−257.77719	−84.10523	3.87018	3.73511
V Order		0.70102	−257.59994	−84.08763	3.47937	3.73518

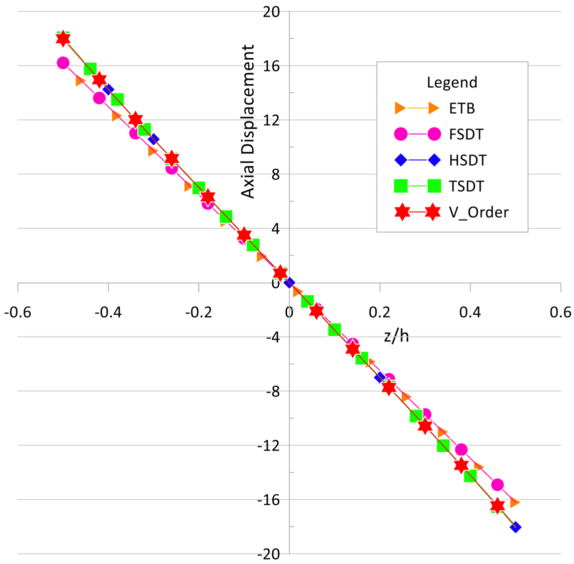

Fig. 3: Variation of axial displacement ($\overline{u}$) through the thickness of beam at (x = 0.25L, z) for aspect ratio 4.

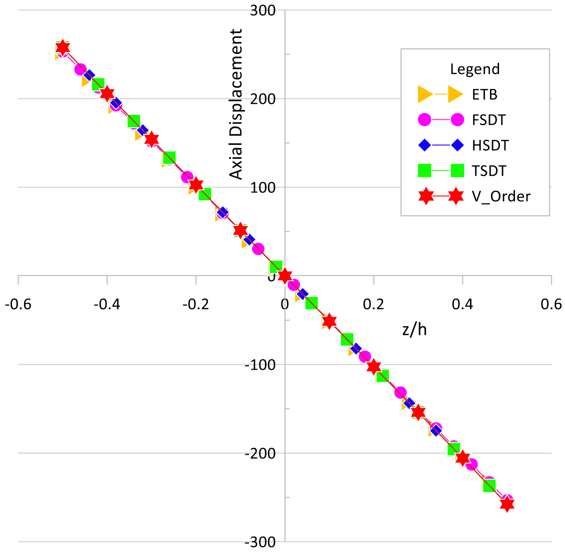

Fig. 4: Variation of axial displacement ($\overline{u}$) through the thickness of beam at (x = 0.25L, z) for aspect ratio 10.

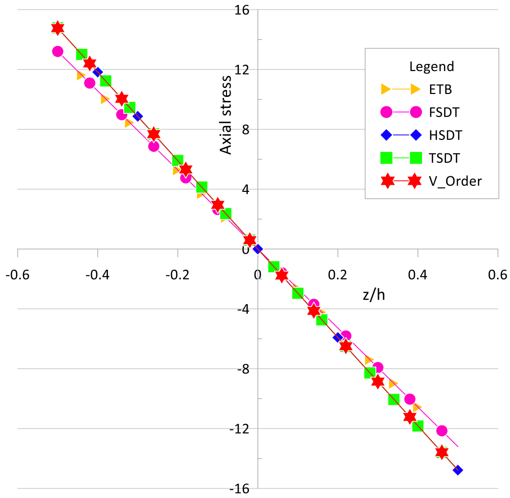

Fig. 5: Variation of axial stress ($\overline{\sigma_x}$) through the thickness of beam at (x = 0.25L, z) for aspect ratio 4.

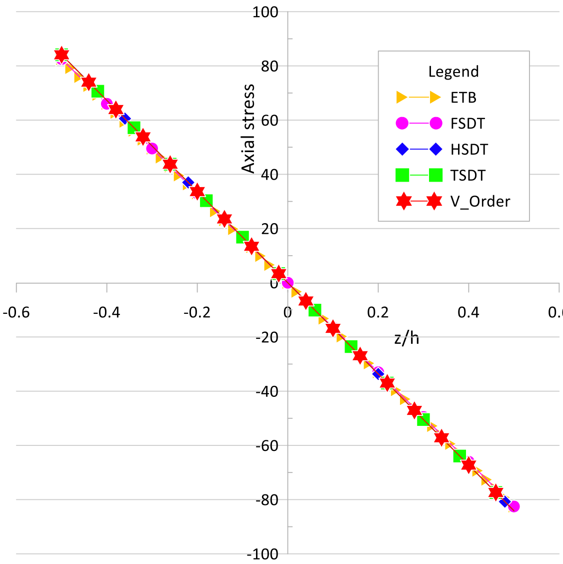

Fig. 6: Variation of axial stress ($\overline{\sigma_x}$) through the thickness of beam at (x= 0.25L, z) for aspect ratio 10.

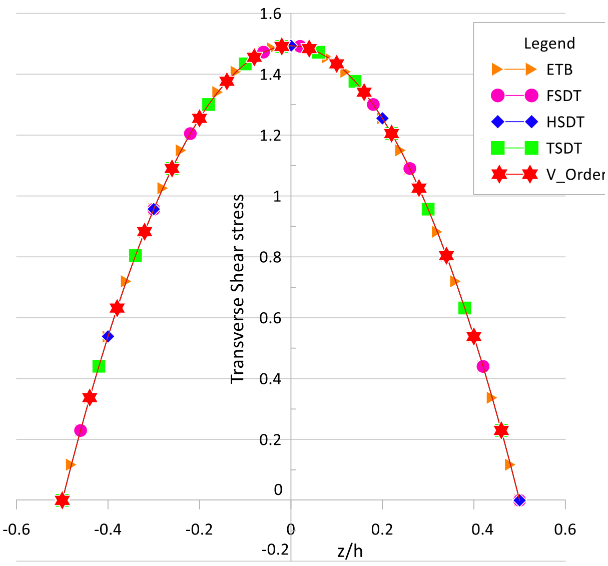

Fig. 7: Variation of transverse shear stress ($\overline{\tau_{zx}}$) through the thickness of beam at (x = 0.25, z) obtain using equilibrium equation for aspect ratio 4.

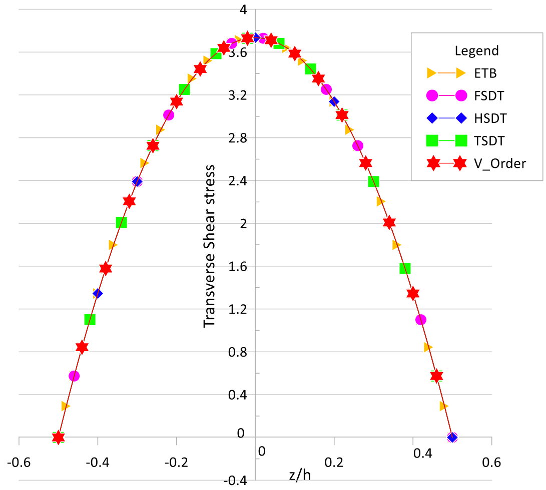

Fig. 8: Variation of transverse shear stress ($\overline{\tau_{zx}}$) through the thickness of beam at (x = 0, z) obtain using equilibrium equation for aspect ratio 10.

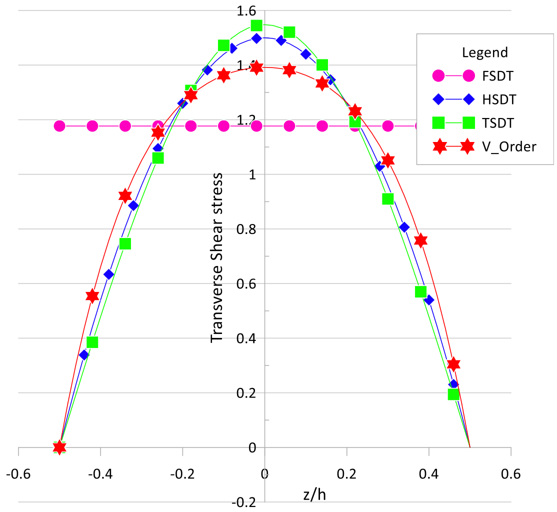

Fig. 9: Variation of transverse shear stress ($\overline{\tau_{zx}}$) through the thickness of beam at (x = 0.25, z) obtain using constitutive relationship for aspect ratio 4.

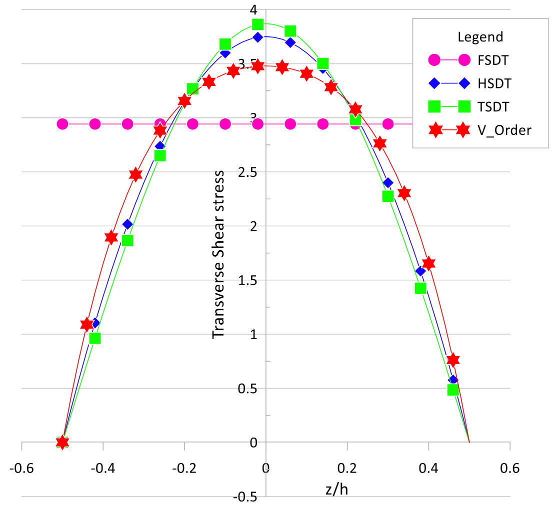

Fig. 10: Variation of transverse shear stress ($\overline{\tau_{zx}}$) through the thickness of beam at (x = 0, z) obtain using constitutive relationship for aspect ratio 10.

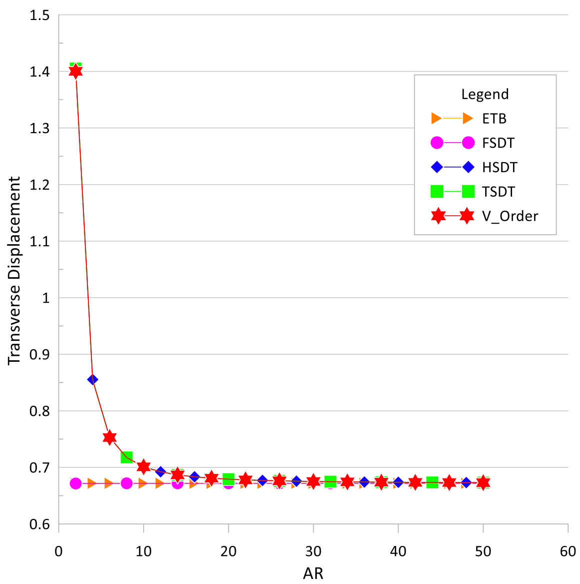

Fig. 11: Variation of transverse displacement ($\overline{w}$) with aspect ratio (S) of beam at (x = 0.25,L, z).

IV. Discussion of Results

Axial Displacement ($\overline{u}$): The comparison of results of maximum non-dimensional axial displacement for the aspect ratios of 4 and 10 is presented in Table 1 and 2. Among the results of all the other theories, the values of axial displacement given by present theory are close agreement with the exact values for aspect ratio 4 and 10. The theories of Bernoulli-Euler as well as Timoshenko for aspect ratio 10, all the refined theories give excellent agreement with each other as compare to exact solution.

Transverse Displacement ($\overline{w}$): Among the results of all the other theories, the values of present theory are in excellent agreement with exact values for aspect ratio 2, 4 and 10. The present theory result are close agreement with exact elasticity solution given by Timoshenko and Goodier.

Axial Stress ($\overline{\sigma_x}$): The axial stress given by present theory are compared with other higher order shear deformation theories, it is observed that result by present theory are in excellent agreement with other theories.

Transverse shear Stress ($\overline{\tau_{zx}}$): The Transverse Shear Stress are obtained by integration of equilibrium equation of two dimensional elasticity. The transverse shear stress satisfies the stress free boundary conditions on the top and bottom surfaces of the beam when these stresses are obtained by both the above mentioned approaches. The comparison of maximum non-dimensional transverse shear stress for a beam with distributed load, obtained by the present theory and other refined theories is presented for aspect ratio of 4 and 10 respectively. FSDT underestimates the value of this stress but use of equilibrium equation gives exact value of this stress.

V. Conclusion

1. The use of present theory gives excellent agreement results available for higher order and refined shear deformation theories for transverse displacement are in tune with the results of present theory.
2. The transverse shear stress when obtained from equilibrium changes its sign but still it is realistic (cosine). The values of transverse shear stress from equilibrium equation are in close agreement.
3. The governing differential equations and the associated boundary conditions are variationally consistent.
4. In general, the use of present theory gives accurate results as seen from the numerical

examples studied and it is capable of predicting the stresses and displacements at support for simply supported beam. This validates the efficacy and credibility of trigonometric shear deformation theory.

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